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A METHOD FOR THE SOLUTION OF STOCHASTIC PROBLEMS IN LINEAR THERMOELASTICITY AND HEAT CONDUCTION*

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Abstract—A method for the determination—in the sense of correlation theory, linear thermoelasticity and heat conduction—of the properties of temperature, stress and displacement in elastic bodies is given. The bodies are stressed by random surface forces and are surrounded by a medium with randomly distributed temperature. Both functions may be random with respect to space and time. The method may be used to derive exact solutions but is perhaps better adaptable for the determination of approximate solutions. One simple example is given.

INTRODUCTION

SPECIAL CASES of the problem of determining the mean values and the correlation functions of temperature, stress and displacement in elastic bodies, stressed by random surface forces and surrounded by a medium with randomly distributed temperature have been treated—always in the sense of linear thermoelasticity—in the literature [1–5].

Parkus [1–3] gives some fundamental relations for temperature and corresponding thermal stresses. The surfaces of the bodies are assumed to be free of tractions, the influence of the coupling between temperature and strain field is ignored. The spatial distribution of the random temperature of the surrounding medium is assumed to be deterministic. In [4, 5] the surface forces and the temperature of the medium surrounding the body are randomly distributed, both with respect to time and space. The influence of coupling and inertia, however, is ignored.

In all examples treated in these papers, exact expressions for the double Laplace transforms of the correlation functions are derived. These expressions are then inverted, exactly or approximately.

Although most of these examples are simple the numerical labor is high. Very often it is impossible to give solutions in the form of tabulated functions [5].

The method described in this paper starts from the variational principle of linear thermodynamics introduced and extensively studied by Biot. It may be used for the determination of exact solutions, usually in the form of infinite series, as well as for the derivation of approximate solutions.

Using the variational principle obtained by Biot from thermodynamical considerations [6–9] the solution of the deterministic problem is given with the aid of Ritz method in Section 1. The equations for expectation and correlation functions are given in Section 2; they are discussed using one simple example of heat conduction in Section 3.

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It should be mentioned that a series of similar methods can be derived in the same manner as described in this paper from other variational principles such as the principles introduced by Parkus [10], Herrmann [11–13], Olszak and Perzyna [14], and Schapery [15].

1. THE DETERMINISTIC PROBLEM

Formulating the variational principle Biot described the thermodynamic state of an elastic body by two vector fields. One is the displacement field $u_i(x_1, x_2, x_3, t)$ of the solid. The other is the field of entropy displacement $S_i(x_1, x_2, x_3, t)$, whose components are the amounts of heat which have flown in given directions divided by the absolute temperature T_0 of the undeformed state, whose existence is postulated. The variation applies to all six components of the two vector fields.

The functions

$$\left.\begin{aligned} {}_mu_i &= \sum_{\alpha=1}^{m} u_{i,\alpha}(x_1, x_2, x_3)\overset{1}{q}_{i,\alpha}(t), \\ {}_nS_i &= \sum_{\alpha=1}^{n} S_{i,\alpha}(x_1, x_2, x_3)\overset{2}{q}_{i,\alpha}(t), \end{aligned}\right\} \tag{1.1}$$

with generalized coordinates $\overset{1}{q}_{i,\alpha}$ and $\overset{2}{q}_{i,\alpha}$ are admissible for the variational principle for suitably chosen functions

$$\left.\begin{aligned} &u_{i,\alpha}(x_1, x_2, x_3), \\ &S_{i,\alpha}(x_1, x_2, x_3) \end{aligned}\right\} \begin{aligned} &i = 1, 2, 3 \\ &\alpha = 1, 2, \ldots \end{aligned}$$

They have to be general enough so that each system u_i and S_i admissible for the variational principle can be approximated by ${}_mu_i$ and ${}_nS_i$ in such a manner that with sufficiently large m, n the values of the functions, the values of the kinetic energy

$$T = \tfrac{1}{2}\int_V \rho \sum^{i} \left(\frac{\partial u_i}{\partial t}\right)^2 \mathrm{d}V, \tag{1.2}$$

and the value of the "dissipation function"

$$D_t = \frac{T_0}{2}\left[\int_V \sum^{ij} \lambda_{ij}\frac{\partial S_i}{\partial t}\frac{\partial S_j}{\partial t}\mathrm{d}V + \int_S \frac{1}{K}\left(\frac{\partial S_n}{\partial t}\right)^2 \mathrm{d}A\right] \tag{1.3}$$

can be approximated to any desired degree of accuracy. The mass per unit volume is denoted by ρ, the coefficient of heat transfer by K, the inverse of the thermal conductivity by λ_{ij} and the component of entropy displacement in the direction of the surface normal n_i (positive outwards) by S_n. The ${}_mu_i$ must satisfy the kinematic boundary conditions with arbitrary $\overset{1}{q}_{i,\alpha}$.

The equations for the generalized coordinates are derived from the Lagrangian equations corresponding to the variational principle:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}}\right) + \frac{\partial D_t}{\partial \dot{q}} + \frac{\partial V_t}{\partial q} = Q. \tag{1.4}$$

Here q stands for $\overset{1}{q}_{i,\alpha}, \overset{2}{q}_{i,\alpha}$, respectively, and Q for the generalized mechanical forces

$$\overset{1}{Q}_{i,\alpha} = \int_S F_i u_{i,\alpha} \, \mathrm{d}A, \tag{1.5}$$

and for

$$\overset{2}{Q}_{i,\alpha} = -\int_S \theta_m n_i S_{i,\alpha} \, \mathrm{d}A, \tag{1.6}$$

respectively. F_i are the components of the surface forces per unit area and $\theta_m = T_m - T_0$, where T_m is the absolute temperature of the surrounding medium. The thermoelastic potential is defined by

$$V_t = \int_V \left(W + \frac{1}{2}\frac{c}{T_0}\theta^2\right) \mathrm{d}V = \int_V \left[W + \frac{1}{2}\frac{T_0}{c}\left(\sum^{i} \frac{\partial S_i}{\partial x_i} + \sum^{ij} \beta_{ij}\varepsilon_{ij}\right)^2\right] \mathrm{d}V, \tag{1.7}$$

where W is the elastic potential

$$W = \tfrac{1}{2} \sum^{ijkl} C_{ijkl}\varepsilon_{ij}\varepsilon_{kl},$$

c the specific heat per unit volume and fixed deformation, θ the temperature increment and β_{ij} are the coefficients of Hooke's law

$$\sigma_{ij} = \sum^{kl} C_{ijkl}\varepsilon_{kl} - \beta_{ij}\theta.$$

Constructing column matrices $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{p}_1$ and $\mathbf{p}_2$, using $\overset{\alpha}{q}_{i,\beta}$ and $\overset{\alpha}{Q}_{i,\beta}$ as elements of $\mathbf{q}_\alpha$ and $\mathbf{p}_\alpha$ with the row number $3(\beta-1)+i$, the kinetic energy and dissipation function are quadratic forms in $\dot{\mathbf{q}}_1$ and $\dot{\mathbf{q}}_2$, respectively, while the thermoelastic potential appears as sum of quadratic forms and a bilinear form in $\mathbf{q}_1$ and $\mathbf{q}_2$.

Employing the symbol $D = \mathrm{d}/\mathrm{d}t$ and the matrices

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_1 \\ \hline \mathbf{q}_2 \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} \mathbf{p}_1 \\ \hline \mathbf{p}_2 \end{bmatrix}, \tag{1.8}$$

the $3m+3n$ linear ordinary differential equations derived from equation (1.4) can be written in the simple form

$$\mathbf{L}(D)\mathbf{q} = \mathbf{p}, \tag{1.9}$$

where $\mathbf{L}(D)$ is a matrix of the form

$$\mathbf{L}(D) = \begin{bmatrix} D^2\mathbf{T} + \mathbf{V}_{11} & \mathbf{V}_{12} \\ \hline \mathbf{V}_{21} & D\mathbf{D} + \mathbf{V}_{22} \end{bmatrix} \tag{1.10}$$

and $\mathbf{V}_{21}$ is the transposed matrix of $\mathbf{V}_{12}$

$$\mathbf{V}_{21} = \tilde{\mathbf{V}}_{12}.$$

Let the body at $t < 0$ be in its undeformed state ($\varepsilon_{ij} = 0, \theta = 0$). The initial values of the generalized coordinates vanish in this case. Denoting the Laplace transform of a function $F(t)$ by $F^*(s)$, the inverse of the matrix $\mathbf{L}$ by $\mathbf{L}^{-1}$ the Laplace transform of $\mathbf{q}$ is

given by

$$\mathbf{q}^*(s) = \mathbf{L}^{-1}(s)\mathbf{p}^*(s). \tag{1.11}$$

Replacement of T in the matrix **L** by the null matrix renders the equations for the quasistatic case.

2. THE RANDOM PROBLEM

If the surface forces or the temperature of the surrounding medium are randomly distributed, the elements of the column matrix **p** are random functions together with the generalized coordinates.

The order of taking expectations, mean square differentiation, m.s. limits and m.s. integration may be interchanged [16, §5.1]. Interpreting differentials, limits and integrals of stochastic functions always in the m.s. sense the equations of Section 1 remain valid if the random functions are replaced by their mean values. Denoting the mean value of a random function x by $\langle x\rangle$, the Laplace transform of the mean value of **q** can be written in the form

$$\langle \mathbf{q}\rangle^* = \mathbf{L}^{-1}(s)\langle \mathbf{p}\rangle^*.$$

The expressions (1.1) are valid for each realization. The corresponding correlation functions of the components of solid and entropy displacement can be represented, accordingly, in the form of double series

$$\left.\begin{aligned}
\langle {}_m u_i(P_1, t_1)\, {}_m u_j(P_2, t_2)\rangle &= \sum^{\alpha\beta} a_{ij,\alpha\beta}(t_1, t_2) u_{i,\alpha}(P_1) u_{j,\beta}(P_2),\\
\langle {}_m u_i(P_1, t_1)\, {}_n S_j(P_2, t_2)\rangle &= \sum^{\alpha\beta} b_{ij,\alpha\beta}(t_1, t_2) u_{i,\alpha}(P_1) S_{j,\beta}(P_2),\\
\langle {}_n S_i(P_1, t_1)\, {}_n S_j(P_2, t_2)\rangle &= \sum^{\alpha\beta} c_{ij,\alpha\beta}(t_1, t_2) S_{i,\alpha}(P_1) S_{j,\beta}(P_2).
\end{aligned}\right\} \tag{2.1}$$

The correlation functions $a_{ij,\alpha\beta}$, $b_{ij,\alpha\beta}$ and $c_{ij,\alpha\beta}$ are the expected values of the products of the generalized coordinates $\overset{1}{q}$ and $\overset{2}{q}$. For instance,

$$a_{ij,\alpha\beta}(t_1, t_2) = \langle \overset{1}{q}_{i,\alpha}(t_1) \overset{1}{q}_{j,\beta}(t_2)\rangle.$$

Accordingly, they are the elements with row number $3(\alpha-1)+i$ and column number $3(\beta-1)+j$ of the matrices $\mathbf{Q}_{11}$, $\mathbf{Q}_{12}$ and $\mathbf{Q}_{22}$ defined by

$$\mathbf{Q}_{ij}(t_1, t_2) = \langle \mathbf{q}_i(t_1) \tilde{\mathbf{q}}_j(t_2)\rangle. \tag{2.2}$$

The correlation function of the Laplace transform of a function $x(t)$ is identical with the double Laplace transform of the correlation function $R_{xx} = \langle x(t_1)x(t_2)\rangle$ of this function [2]

$$R^{**}_{xx}(s_1, s_2) = \langle x^*(s_1)x^*(s_2)\rangle \equiv \mathscr{L}^2\{R_{xx}(t_1, t_2)\}.$$

The elements with row number $3(\alpha-1)+i$ and column number $3(\beta-1)+j$ of the matrices $\mathbf{P}_{11}$, $\mathbf{P}_{12}$ and $\mathbf{P}_{22}$ defined by

$$\mathbf{P}_{ij}(t_1, t_2) = \langle \mathbf{p}_i(t_1) \tilde{\mathbf{p}}_j(t_2)\rangle, \tag{2.3a}$$

are the integrals

$$\left.\begin{aligned}
&\int_S\int_S \langle F_i(P_1,t_1)F_j(P_2,t_2)\rangle u_{i,\alpha}(P_1)u_{j,\beta}(P_2)\,\mathrm{d}P_1\,\mathrm{d}P_2,\\
&\int_S\int_S \langle F_i(P_1,t_1)\theta_m(P_2,t_2)\rangle u_{i,\alpha}(P_1)n_j(P_2)S_{j,\beta}(P_2)\,\mathrm{d}P_1\,\mathrm{d}P_2,\\
&\int_S\int_S \langle \theta_m(P_1,t_1)\theta_m(P_2,t_2)\rangle n_i(P_1)S_{i,\alpha}(P_1)n_j(P_2)S_{j,\beta}(P_2)\,\mathrm{d}P_1\,\mathrm{d}P_2.
\end{aligned}\right\} \tag{2.3b}$$

The matrix $\mathbf{Q}$

$$\mathbf{Q}(t_1,t_2) = \langle \mathbf{q}(t_1)\tilde{\mathbf{q}}(t_2)\rangle = \left[\begin{array}{c|c} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \hline \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{array}\right] \tag{2.4}$$

and its partitioned matrices $\mathbf{Q}_{11}$, $\mathbf{Q}_{12}$ and $\mathbf{Q}_{22}$ can be determined with the aid of the matrix

$$\mathbf{P}(t_1,t_2) = \langle \mathbf{p}(t_1)\tilde{\mathbf{p}}(t_2)\rangle = \left[\begin{array}{c|c} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \hline \mathbf{P}_{21} & \mathbf{P}_{22} \end{array}\right] \tag{2.5}$$

and equation (1.11), valid for each realization, from the equation

$$\mathbf{Q}(t_1,t_2) = \mathscr{L}^{-2}\{\langle \mathbf{q}^*(s_1)\tilde{\mathbf{q}}^*(s_2)\rangle\} = \mathscr{L}^{-2}\{\mathbf{L}^{-1}(s_1)\mathbf{P}^{**}(s_1,s_2)\tilde{\mathbf{L}}^{-1}(s_2)\}. \tag{2.6}$$

If the input of a damped system is a stationary process this is, after a sufficiently long time, also true for the output. The simplest way for the determination of the correlation functions for large time is via the spectral densities S.

The Wiener–Chintschin formulas relate the spectral densities with their correlation functions. Let $\mathscr{F}_c\{F(t)\}$ denote the Fourier cosine transform of a real function $F(t)$

$$\mathscr{F}_c\{F(t)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty F(t)\cos\omega t\,\mathrm{d}t.$$

With $\tau = |t_2 - t_1|$ these formulas can be written in the form

$$S_{xx}(\omega) = \mathscr{F}_c\{R_{xx}(\tau)\}, \qquad R_{xx}(\tau) = \mathscr{F}_c\{S_{xx}(\omega)\}. \tag{2.7}$$

The spectral density of an input $x(t)$ is related to the spectral densities of two outputs $y(t)$ and $z(t)$ by

$$\left.\begin{aligned}
S_{yy}(\omega) &= \overline{H_y(\omega)}H_y(\omega)S_{xx}(\omega),\\
S_{yz}(\omega) &= \overline{H_y(\omega)}H_z(\omega)S_{xx}(\omega),
\end{aligned}\right\} \tag{2.8}$$

where $H_y(\omega)$ and $H_z(\omega)$ are the frequency response functions of $y(t)$ and $z(t)$, respectively, and $\overline{H(\omega)}$ denotes the conjugate complex of $H(\omega)$.

For the values of the matrix $\mathbf{Q}$ for large time, one obtains the simple equation

$$\mathbf{Q}(\tau) = \mathscr{F}_c\{\mathbf{L}^{-1}(-i\omega)\mathbf{S}(\omega)\tilde{\mathbf{L}}^{-1}(i\omega)\}, \tag{2.9}$$

where the notation

$$\mathbf{S}(\omega) = \mathscr{F}_c\{\mathbf{P}(\tau)\} \tag{2.10}$$

has been used.

With the aid of the correlation functions $a_{ij,\alpha\beta}$, $b_{ij,\alpha\beta}$ and $c_{ij,\alpha\beta}$ the correlation functions of temperature, deformation and stress can be determined too.

If the surface forces and the temperature of the surrounding medium are linearly independent and if one of the mean values is zero, the elements of the matrices $\mathbf{P}_{12}$ and $\mathbf{P}_{21}$ vanish. In addition, the elements of the corresponding matrix $\mathbf{P}_{11}$ or $\mathbf{P}_{22}$ vanish in the case of vanishing surface forces or vanishing medium temperature. The equations simplify then considerably.

3. EXAMPLE

The example given in [4] of determining the covariance of the temperature in an infinite, homogeneous and isotropic circular cylinder shall now be treated using the method described above. The cylinder of radius R with its axis coinciding with x_3 is initially at zero temperature. From time $t = 0$ on, the random temperature θ_0, independent of the coordinate x_3, is imposed on the lateral surface of the cylinder.

For each cross-section $x_3 = \text{const.}$, lying in the finite, we have a two-dimensional problem of heat conduction. Using circular-cylindrical coordinates $r, \vartheta, z \equiv x_3$, the covariance of the temperature of two points on the surface is assumed to be given by

$$C \exp(-\kappa_1 r_{12}^2 - \kappa_2 |t_2 - t_1|), \tag{3.1}$$

where r_{12} denotes the distance of the points $P_1(R, \vartheta_1)$ and $P_2(R, \vartheta_2)$

$$r_{12}^2 = \left[2R \sin \frac{\vartheta_2 - \vartheta_1}{2}\right]^2.$$

It has been noted by Biot [17] that the number of generalized coordinates necessary for the determination of the temperature can be reduced considerably using special systems of functions. The equations of these generalized coordinates decouple from each other and also from the remaining equations necessary for the calculation of the entropy displacement.

Using the expansion

$$S_i = \frac{k}{T_0} \sum^{\alpha} \frac{1}{\lambda_\alpha} \frac{\partial \theta_\alpha}{\partial x_i} q_\alpha \tag{3.2}$$

where λ_α are the eigenvalues and θ_α the eigenfunctions of the eigenvalue problem

$$\lambda_\alpha \theta_\alpha + a \nabla^2 \theta_\alpha = 0 \tag{3.3}$$

with boundary condition

$$K\theta_\alpha + k \sum^{i} n_i \frac{\partial \theta_\alpha}{\partial x_i} = 0 \tag{3.4}$$

the temperature can be represented by the series

$$\theta = \sum^{\alpha} \theta_\alpha q_\alpha. \tag{3.5a}$$

a is the thermal diffusivity, $a = k/c$.

In the problem under consideration boundary condition (3.3) has to be replaced by

$$\theta_\alpha = 0. \tag{3.5b}$$

The normalized eigenfunctions θ_α of this problem are [18, p. 254]

$$\sqrt{\frac{\varepsilon_n}{\pi}\frac{J_n(j_{n,\alpha}\rho)}{RJ'_n(j_{n,\alpha})}}\begin{cases}\cos n\vartheta \\ \sin n\vartheta,\end{cases} \tag{3.6}$$

where $\rho = r/R$. J_n denotes the Bessel function of the first kind with $j_{n,\alpha}$ as its zeros

$$J_n(j_{n,\alpha}) = 0.$$

$J'_n(z)$ indicates derivatives with respect to the argument

$$J'_n(z) = \frac{\mathrm{d}J_n(z)}{\mathrm{d}z},$$

ε_n is the Neumann factor

$$\varepsilon_n = \begin{cases}1 & n = 0 \\ 2 & n \neq 0.\end{cases}$$

Due to the orthogonality of the eigenfunctions $\mathbf{V}_{22}$ is a diagonal matrix. The eigenfunctions are normalized and

$$V_t = \frac{1}{2}\frac{c}{T_0}\int_V \theta^2 \,\mathrm{d}V. \tag{3.7}$$

The values of the diagonal elements are therefore

$$\frac{c}{T_0}\int_0^R\int_0^{2\pi} \theta_\alpha^2 r \,\mathrm{d}r\,\mathrm{d}\vartheta = \frac{c}{T_0}. \tag{3.8}$$

The eigenfunctions diagonalize also the matrix $\mathbf{D}$. The diagonal elements can be calculated easily remembering that $\theta_\alpha \exp(-\lambda_\alpha t)$ is a solution of the heat conduction equation with vanishing temperature. This immediately yields the diagonal elements of $\mathbf{D}$:

$$d_{\alpha\alpha} = \frac{c}{T_0\lambda_\alpha} \tag{3.9}$$

The elements of the matrix $\mathbf{P}_{22}^{**}$ can be calculated easily with the aid of the residue theorem. Introducing Kronecker's symbol δ_{mn} one has

$$\int_0^{2\pi}\int_0^{2\pi} \exp(-\kappa_1 r_{12}^2)\cos m\vartheta_1 \cos n\vartheta_2 \,\mathrm{d}s_1\,\mathrm{d}s_2 = \frac{4\pi^2R^2}{\varepsilon_n}\exp(-2\kappa_1R^2)I_n(2\kappa_1R^2)\delta_{mn},$$

$$\int_0^{2\pi}\int_0^{2\pi} \exp(-\kappa_1 r_{12}^2)\sin m\vartheta_1 \sin n\vartheta_2 \,\mathrm{d}s_1\,\mathrm{d}s_2 = \frac{4\pi^2R^2}{\varepsilon_n}\exp(-2\kappa_1R^2)I_n(2\kappa_1R^2)\delta_{mn} \quad n \neq 0,$$

where I_n are the modified Bessel functions of the first kind. The integrals

$$\int_0^{2\pi}\int_0^{2\pi} \exp(-\kappa_1 r_{12}^2)\cos m\vartheta_1 \sin n\vartheta_2 \,\mathrm{d}s_1\,\mathrm{d}s_2, \qquad \int_0^{2\pi}\int_0^{2\pi} \exp(-\kappa_1 r_{12}^2)\sin m\vartheta_1 \cos n\vartheta_2 \,\mathrm{d}s_1\,\mathrm{d}s_2$$

vanish.

With the abbreviations

$$\bar{C} = C\frac{4a^2}{R^4}\exp(-2\kappa_1 R^2), \qquad C_n = \varepsilon_n I_n(2\kappa_1 R^2), \tag{3.10}$$

$$F_\alpha(\rho) = \frac{j_{n,\alpha}J_n(j_{n,\alpha}\rho)}{J_{n-1}(j_{n,\alpha})}, \tag{3.11a}$$

$$G_{\alpha\beta}(t_1, t_2) = \frac{2\kappa_2}{(\kappa_2^2-\lambda_\beta^2)(\lambda_\alpha+\lambda_\beta)}\exp[-\lambda_\beta(t_2-t_1)]$$

$$+\frac{1}{(\lambda_\alpha+\kappa_2)(\lambda_\beta-\kappa_2)}\exp[-\kappa_2(t_2-t_1)]+\frac{\lambda_\alpha+\lambda_\beta-2\kappa_2}{(\kappa_2-\lambda_\alpha)(\kappa_2-\lambda_\beta)(\lambda_\alpha+\lambda_\beta)}\exp(-\lambda_\beta t_2-\lambda_\alpha t_1) \tag{3.11b}$$

$$+\frac{1}{(\kappa_2-\lambda_2)(\kappa_2+\lambda_\beta)}\exp(-\kappa_2 t_1-\lambda_\beta t_2)+\frac{1}{(\kappa_2-\lambda_\beta)(\kappa_2+\lambda_\alpha)}\exp(-\kappa_2 t_2-\lambda_\alpha t_1),$$

the covariance of the temperature for $t_2 \geq t_1$ is finally given* by the series

$$\bar{C}\sum_{n=0}^{\infty} C_n \cos n(\vartheta_2-\vartheta_1)\sum^{\alpha\beta} F_\alpha(\rho_1)F_\beta(\rho_2)G_{\alpha\beta}(t_1, t_2). \tag{3.12}$$

This expression vanishes at $r = R$ and, therefore, does not converge uniformly in the region $r \leq R$. To evaluate it for points close to the boundary with satisfactory accuracy a great number of terms has to be retained. Frequently, however, (3.12) can be transformed into a well converging series. For instance, putting $P_1 = P_2$ and $t_1 = t_2 = t$ the variance can be represented in the special case $\kappa_2 = 0$ and $(a/R^2)t \gg 1$ by the uniformly converging series

$$C\exp(-2\kappa_1 R^2)\sum_{n=0}^{\infty}\varepsilon_n I_n(2\kappa_1 R^2)\rho^{2n}$$

using [4, p. 302]

$$2\sum^{\alpha}\frac{J_n(j_{n,\alpha}\rho)}{j_{n,\alpha}J_{n-1}(j_{n,\alpha})} = -\rho^n. \qquad \rho < 1$$

This series agrees with the series obtained in [4].

REFERENCES

[1] H. PARKUS, Temperaturfelder bei zufallsabhängiger Oberflächentemperatur. *Anz. öst. Akad. Wiss.* **98**, 153 (1961).

[2] H. PARKUS, Wärmespannungen bei zufallsabhängiger Oberflächentemperatur. *Z. angew. Math. Mech.* **42**, 499 (1962).

[3] H. PARKUS, Methods of solution of thermoelastic boundary value problems. *Proc. 3rd Symp. on Naval Structural Mechanics*, pp. 317–347. Pergamon Press (1963).

[4] J. L. ZEMAN, Örtlich und zeitlich zufällig verteilte Temperatur- und Spannungsfelder, Erster Teil. *Acta mech.* **1**, 194 (1965).

[5] J. L. ZEMAN, Örtlich und zeitlich zufällig verteilte Temperatur- und Spannungsfelder, Zweiter Teil. *Acta mech.* **1**, 371 (1965).

[6] M. A. BIOT, Theory of stress–strain relations in anisotropic viscoelasticity and relaxation phenomena. *J. appl. Phys.* **25**, 1385 (1954).

* Using Voelker and Doetsch [19], Korr. A2–22, A2–76 and A4–42.

[7] M. A. BIOT, Variational principles in irreversible thermodynamics with application to viscoelasticity. *Phys. Rev.* **97**, 1463 (1955).
[8] M. A. BIOT, Thermoelasticity and irreversible thermodynamics. *J. appl. Phys.* **27**, 240 (1956).
[9] M. A. BIOT, Linear thermodynamics and the mechanics of solids. *Proc. 3rd U.S. Natn. Congr. Appl. Mech.*, pp. 1–18. ASME (1958).
[10] H. PARKUS, Über eine Erweiterung des Hamiltonschen Prinzipes auf thermoelastische Vorgänge, Beiträge zur angewandten Mechanik (Federhofer-Girkmann-Festschrift), pp. 295–301. Deuticke, Wien (1950).
[11] G. HERRMANN, On a complementary energy principle in linear thermoelasticity. *J. Aero/Space Sci.* **25**, 660 (1958).
[12] G. HERRMANN, Energy methods for the analysis of temperature distributions and thermal stresses in structures. Prelim. publ. *6th Congr. Int. Ass. Bridge & Struct. Engrs.*, Leemann, Zürich, 1960.
[13] G. HERRMANN, On variational principles in thermoelasticity and heat conduction. *Q. appl. Math.* **21**, 151 (1963).
[14] W. OLSZAK and P. PERZYNA, Variational theorems in general viscoelasticity. *Ing.-Arch.* **28**, 246 (1959).
[15] R. A. SCHAPERY, Irreversible thermodynamics and variational principles with applications to viscoelasticity. Aeronautical Research Laboratories, Office of Aerospace Research, 62–418 (Aug. 1962).
[16] M. S. BARTLETT, *An Introduction to Stochastic Processes*. Cambridge University Press (1960).
[17] M. A. BIOT, Further developments of new methods in heat-flow analysis. *J. Aero/Space Sci.* **26**, 367 (1959).
[18] R. COURANT and D. HILBERT, *Methoden der Mathematischen Physik*, Bd. 1. Springer (1924).
[19] D. VOELKER and G. DOETSCH, *Die zweidimensionale Laplace-Transformation*. Birkhäuser (1950).

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Résumé—Une méthode pour la détermination des propriétés de température, de travail et de déplacement dans les corps élastiques y est donnée en fonction d'une théorie de corrélation de la thermo-élasticité linéaire et de la conduction de la chaleur. Les corps sont soumis au travail au moyen de forces superficielles quelconques et sont environnés par un milieu de distribution de température quelconque également. Les deux fonctions sont de tout-venant tant soit pour le temps que pour l'espace. La méthode peut être utilisée pour en dériver des solutions exactes mais pourrait possiblement être mieux adaptée à la déterminations de solutions approximatives. Un exemple simple y est donné.

Zusammenfassung—In der vorliegenden Arbeit wird eine Methode zur Bestimmung der korrelationstheoretischen Eigenschaften des Temperatur-, des Spannungs- und des Verschiebungsfeldes in elastischen Körpern, die durch zufallsabhängige Oberflächenkräfte beansprucht werden und die sich in einem Medium mit zufallsabhängiger Temperatur befinden, im Rahmen der linearen Thermoelastizität angegeben. Die Zufallsfunktionen, die Oberflächenkräfte und die Umgebungstemperatur, können dabei sowohl örtlich wie auch zeitlich zufällig verteilt sein. Die Methode ist geeignet zur Bestimmung korrelationstheoretisch exakter Lösungen und zur Herleitung von Näherungslösungen. Ein einfaches Beispiel wird angegeben.

Абстракт—Даётся метод для определения—в смысле теории соотношения (корелляции) линейной термозластичности и проводимости тепла—свойств температуры, напряжения, и смещения в упругих телах. Тела подвергаются напряжению слугайных поверхностных сил и окружены средой со случайно распредённой температурой. Обе функции могут быть случайными, принимая во внимание пространство и время. Метод может применяться для того, чтобы вывести точные решения, но, может быть более приемлем для определения приблизительных решений. Дается один простой пример.